Block Matrices Linear Algebra X. Du

- Block matrices are interpreted as being partitioned into submatrices
- We have already seen some of these partitions in Gaussian elimination and computing the reduced row-echelon form.
- Another common form of block matrices is partitioned by rows or columns.
- Multiplication is done normally, except you can break down the process with blocks.
    - For example, let A and B be block matrices of the form $A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. Then $AB=\begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{pmatrix}$
    - This can be useful if a block is an identity matrix or a zero matrix.
- Inversion
    - If a matrix block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square blocks. Then $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$.
- **Direct sum**
    - $A\oplus B=\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where 0 represents a block of zeros
    - Example: $A=\begin{pmatrix} 1 & 7 \\ 9 & 3 \end{pmatrix}, B=\begin{pmatrix} 4 & 0 & 6 \end{pmatrix}$. $A\oplus B=\begin{pmatrix} 1 & 7 & 0 & 0 & 0 \\ 9 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 6 \end{pmatrix}$
- **Block diagonal matrix**
    - Square matrix with the following form:
        - $A=\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}$
        - $A_1, A_2, ..., A_n$ are square matrices. 0 represents a block of zeros.
    - $A = A_1 \oplus A_2 \oplus ... \oplus A_n$
    - $\det(A)=\det(A_1)\det(A_2)...\det(A_n) \qquad tr(A)=tr(A_1)+tr(A_2)+...+tr(A_n)$
    - $A^{-1}=\begin{pmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^{-1} \end{pmatrix}$
    - Set of eigenvalues and eigenvectors of A is the union of those of $A_1, A_2, ..., A_n$